The co-emergence of sets and functions in Das Kontinuum

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Symposium "Weyl épistémologue", 15 décembre 2005

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in Hermann Weyl's *Das Kontinuum*

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The chapter I of *Das Kontinuum* invites us to rethink, in a radical way, the nature of the relations between the two main categories of mathematical entities: sets and functions. An outstanding point of this reform is what I called the **"co-emergence"** of sets and functions within the mathematical universe. In other words, none of these two types of entities come first in the genesis of this universe. Sets and functions cover the totality of it and came from the same "mathematical process".

? *What does this co-emergence mean?*

- ? *What can we learn from it about the nature of the relations between the notion of set and the notion of function in Das Kontinuum?*

To make explicit Weyl's position and to test its consistency, we will study the nature of sets and functions from a strictly *logical and mathematical* point of view.

In this way, we are putting deliberately the *continuum problem* aside. Nevertheless, the continuum problem constitutes the heart of Weyl's project and gives to the mathematical construction his pertinence, submerging it in the theoretical project of physics. In this way, the arithmetical construction of the continuum is justified from the outside and on the whole. However, the internal point of view on the first chapter of *Das Kontinuum* is legitimate in the opinion of Weyl himself[1].

Thus, let's take this mathematical and logical point of view, and let's return to the co-emergence of sets and functions. This feature of Weyl's system seems to have not been very studied. In fact, the first chapter of *Das Kontinuum* is often studied by the means of axiomatic reconstructions. This method permits to have a comprehensive view on the system and permits to make a direct comparison with similar systems. However, all the axiomatic reconstructions I know conceal partly the common nature of sets and functions in this system. For example, the translation of Weyl's system in usual notations often compels us to replace the single mathematical process by several comprehension axioms (one for sets and one for functions). Thus, without really betraying Weyl's system, these reconstructions are inadequate for the understanding of the specificity of Weyl's notion of function and of its particular relations to the notion of set.

But Weyl's conception of the relations between the notions of function and set merits our attention. Indeed, Weyl turns upside down the order between the notion of function and the notion of set that is the one of Set Theory. Our comparison with Set Theory is not arbitrary. Since it is not only the theory that is used as a basis for numerous mathematical disciplines today but this theory is also the starting point of Weyl's thought on the mathematical foundations[2].

[1] [1918], Preface, p1.
[2] [1918], p47

Let's remind us that, in Set Theory, a function is nothing but a *particular kind of set*. A function is likened to its graph. A function from X to Y is a particular *subset* of the product X*Y. "particular" means that this subset must verify the property of uniqueness of the "y" for each "x".

In fact, Set Theory use another notion of function; that which is used, for example, in the wording of Fraënkel's replacement axiom for the axiomatic theory ZF. In this other sense, a function is a two-arguments predicate whose arguments are not restricted to two sets X and Y. A function, in this sense, can apply to *any* set in order to define another set. But such a function is only an *intensional* entity to which we cannot assign any extension[3]. A function, in this sense, is not really an object from the mathematical universe of Set Theory but it is rather a definitional entity that takes part in the definition of sets.

If we pay attention only to the *extensional* notion of function, the functions are thus *kinds of set*. We don't have to be surprised by this result since Set Theory defends an *ontological monism*. The universe of Set Theory seems to be plentiful but all its entities must be placed under one single kind: they are all "sets". Functions are ontologically derived. That is, their admission in the mathematical universe needn't axioms in addition to those that are given for the existence of sets.

In the opposite direction, sets and functions spring up simultaneously in Weyl's system. The *mathematical process* stands in for the comprehension axiom. It permits to assume simultaneously sets and functions in the mathematical universe. It is what I called "co-emergence". However, we don't have an ontological dualism since sets and functions are defined so that sets become kinds of functions. The inclusion between sets and functions is therefore turned upside down in comparison to Set Theory. We have still an ontological monism but this is not this time a Set Theory but rather a *function theory*.

The link between these two notions is however stronger in Weyl's system. Indeed, the notion of set is integral part of the notion of function.

[3] This assertion would have to be qualified if we considered the situation from the point of view of Classes theory, for example that of Bernays and Von Neumann. From this point of view, the kind of functions intervening in the replacement axiom is regarded as a kind of class. Their status is then ambiguous. A class is an extensional entity only in a weak sense. Most of the operations allowed for sets are forbidden for classes. This point is not however very important according to our intention??.

Thus, the notion of function is an *extension* of the notion of set. On the contrary, in Set Theory, none of the two notions were the extension of the other.

After having emphasized the originality of the relations of the notions of set and function in Weyl's position, we propose carrying out two tasks:

1) First, we will lighten the meaning of this "co-emergence" of sets and functions in Weyl's system, making explicit the reasons for their having a common nature and for the origin of their distinction.

2) Secondly, we will explain why Weyl was compelled to assume, in his mathematical universe, these odd entities that he called « functions » and whose notion is the origin of the turning upside down we have expounded.

Owing to the reasons given above, we will try to complete these tasks analysing how *Weyl himself* sets out his system in the first chapter of *Das Kontinuum* and in a letter to Hölder[4].

In order to complete the first task, we must first outline Weyl's position in 1918 about the (internal) foundations of mathematics. We will base our sketch on the text of *Das Kontinuum,* on two articles of Solomon Feferman, and on personal thoughts.

This position can be sketched by four terms: definitionism, intuitionism, predicativism, and arithmetism. Let's explain what those terms mean.

Definitionism

This position consists in refusing to assume the insertion of a new ideal object (that is a set or a function) if it is not introduced by the way of giving explicitly the relation which links together the constituent elements of this ideal object. We can express it in a different way asserting that each extensional entity must be introduced by the means of an intensional entity.

[4] [1919]

Intuitionism

Secondly, his mathematical universe is restricted in the sense that all the entities assumed must be generated by the logical principles from the a *basic category of entities* which are given intuitively. This basic category is a very *structure* made up of *primitive* objects and relations. The intuitive knowledge we must have of those entities must give foundations to the Excluded-Middle Principle. Each rightly built proposition, which concerns only the primitive entities, admits one truth-value, regardless our ability to determine it. Weyl expresses it asserting that such a category is a "complete system of definite self-existent objects". This kind of "intuitionism" is therefore far from Brouwer's one to which Weyl adhered a few years later.

Predicativism

Weyl refuses every *impredicative* definition, that is every definition which supposes the prior given information of a totality of entities of which the object to define is one of the members. Weyl's predicativism is expressed by the "*restricted principle".* In a language which is not that of Weyl, it consists in restricting the scopes of the quantifiers to the primitive entities. This principle permits to eliminate impredicative definitions while it expresses the privileged access we have to the basic categories. In the language of the Ramified Type Theory(RTT), introduced by Russell and Whitehead in the *Principia Mathematica*, the restricted principle consists in assuming only the entities of level 1.

The logical principles

Those three major thesis of Weyl's position (definitionism, intuitionism and predicativism) gives rise to the formulation of 6 logical principles which can be indifferently interpreted as principles of the construction of propositions or as principles of the construction of relations.

There are six principles:

1) The negation principle
2) The blanks-identification principle
3) The conjunction principle
4) The disjunction principle
5) The "filling in" principle
6) The "there is" principle[5]

[5] [1918], p9-10

We don't give details of those principles. Let's remind us solely that they permit to construct each property (or relation) which can be expressed in the first-order-predicates language, from the primitives relations, and including symbols for sets, functions and for the membership relation Є.

Arithmetism : natural numbers and iteration

The completeness condition, which Weyl demands from the basic categories, is a very strong one. Owing to this fact, Weyl develops only one such category: that of natural numbers.

The natural numbers series is important in Weyl's position because it makes up the intuitive datum which permits the foundation of a new type of definition of relation (the principle of iteration) and therefore a new form of inference(the inference by complete induction).

Two types of intuition are linked to the natural numbers series.

1) We have the *intuition* that gives us the natural numbers series *as a complete system*.

2) We have the *intuition of the iteration*. We mean here the intuition by which we can assert that, when we have an homogeneous relation (that is a relation which links each object to another object of the same nature), we can then consider the iteration of this operation an indefinite number of times.

Those two intuitions are linked together because:

1) The category of natural numbers can be regarded as the totality of the elements obtained by the *iteration* of the "successor relation" from the number 1.

2) The completeness of the natural numbers series exports itself to the outside, in a way, by the means of the intuition of iteration. Let's take an homogeneous one-to-one relation R(x,y) (x and y belong to *any* same category) Then, the totality of the elements obtained by the successive iterations of the relation "R" must be regarded as a *complete* system just like the natural numbers series itself.

Those two intuitions (that of the « completeness » of the natural numbers series, and that of the possibility to repeat indefinitely the iteration of an homogeneous operation) are blending together so that Weyl seems to confuse them entirely:

> **«[…]the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought »[6]**

The second feature we have expounded is an essential one for Weyl's system. I choose the term "transcendence" to designate this distinctive feature of Weyl's theory by which the completeness of natural numbers series exports itself outside. This property of Weyl's system shows that he had, in a way, a *formal* conception of natural numbers in spite of his rejection of a *formalist* opinion on mathematics. His conception of natural numbers is a *formal* one in the sense that what is important in the natural numbers series is its *structure*. But it is not a *formalist* opinion because Weyl didn't think at all that this structure emerges from an arbitrary choice. We have to remind that the natural numbers series is given intuitively to us, and is given in such a way that we know it is a *complete* totality and we know that its completeness exports itself to other series. We are then far from a formalist position.

(We can't here develop all the arguments we have to show that Weyl's notion of the natural series is a in a way a *formal* one. The main arguments are:

- Weyl's assimilation of the idea of the iteration and of the idea of the natural numbers series.
- Weyl's refusal to base the natural numbers series on the definition of an essence for each isolated natural number. The only satisfactory way to define a number is to give is place in the succession.[7]
- Weyl's agreement with an axiomatic point of view on the natural numbers series (providing that we give their real status to axioms).
- The fact that, in *Das Kontinuum,* nothing is told about the nature of natural numbers but in relation with their succession.
- The particular form of the principle of iteration wich assumes Weyl. Cf above)

[6] [1918], p48

[7] cf [1918] p47 for the refusal of Frege's and Russell's trial to define each natural number as an equivalent classes. Cf [1949] I. 6, p34 where Weyl asserts that "the *succession* of numbers appears as their constitutive characteristic".

The principle of iteration

This principle is the principle of construction that expresses, *at the logical level*, what we have called the transcendence of the natural numbers series. According to this principle, a totality of *sets* obtained by successive iterations of an homogeneous set-operation Φ(X) can be regarded as a complete totality, available for the definition of a new entity.

The schema below illustrate the meaning of this principle in the simplest case.

a)

$$\underbrace{1 \xrightarrow{\text{successor function}} 2 \xrightarrow{\text{successor function}} 3 \quad \ldots}$$

This totality is complete. We can thus use, in order to define a new entity, a relation like "there is a natural number n such as…"

$\Big\Downarrow$ **Transcendence of the natural numbers series**

b)

$$\underbrace{X_1 \xrightarrow{\text{homogeneous function } \Phi} X_2 \xrightarrow{\text{homogeneous function } \Phi} X_3 \quad \ldots}$$

This totality is complete. We can thus use, in order to define a new entity, a relation like "there is a *set* among (X_1, X_2, …) such as…"

Explanation of the relations between sets and functions in *Das Kontinuum*

Having reminded the main thesis of Weyl about the foundations of mathematics, we can now carry out the two tasks we had proposed to us. To explain the relations between sets and functions we have to study the nature of the transition from intension to extension.

The mathematical process

We have already referred to the fact that this transition works, for sets and functions, by the same way: the process that Weyl calls "mathematical process". Let's see how it works.

Let's assume we have a relation R(, , …) that is constructed in accordance to Weyl's principles and which can therefore be used for the definition of an ideal entity. (this kind of relation is called "finitist" by Weyl). Each argument is linked to a category which can be basic *or not* (a category of sets, etc.) We have to suppose that the arguments are divided in two groups: the *dependent ones* and the *independent* ones.

$$R\,(\underbrace{x_1,\ \dots\ ,\ x_k}_{\text{Dependent Variables}} \mid \underbrace{y_1,\ \dots\ ,\ y_n}_{\text{Independent Variables}})$$

Case of sets

Let's talk firstly of the case n=0, that is the case where there is no independent variable. Then, the mathematical process links to the relation R (x_1 , … , x_k) the k-dimensional set Ř. Two factors step in this transition from the relation to the set:

1) The variables disappear. In Frege's way of speaking, the set Ř is a saturated entity on the contrary to the relation from which it is constructed.

2) The identification criterion changes. Let's quote Weyl :

« Therefore, how two *sets* […] are defined […] does not determinate their identity [on the contrary to *relations*] Rather an *objective* fact, which is not decidable from the definition in a purely logical way, is decisive; namely, whether each element of the one set is also an element of the other, and conversely. »

Case of functions

Let's assume now that $n \neq 0$, that is there is at least one independent variable. Then, the mathematical process links to the relation $R(x_1, \ldots, x_k \mid y_1, \ldots, y_n)$ the *function* $\check{R}(y_1, \ldots, y_n)$. For each possible value of the arguments, the function $\check{R}(y_1, \ldots, y_n)$ becomes a set. This transition from the relation to the function includes two factors:

1) One part of the variables disappears (the dependent ones).

2) The identification criterion changes in a similar way to the case of sets. Two functions are identical if their values are identical for each possible determination of the variables.

Thus, We find again the same two factors.

Before giving details about it, let's make two remarks.

First, we can see immediately the difference between *this* notion of function and the *set-theoretical* one. Indeed, in Set Theory, the value of a function can be of any nature whereas in Weyl's system the value of a function can't be a basic object. In fact, to express a one-to-one relation between basic objects, the notion of *set* (in Weyl's sense) is enough. (such a relation can be rendered in Weyl's system by a two-dimensional set $\check{R}$ which verify the property that for each x there is a y such as « $(x,y) \in \check{R}$ ». this property is a finitist one)

Secondly, we can see now the exact relation between the notions of sets and functions in *Das Kontinuum*. We remark that the notion of function is defined by the means of the notion of set. Sets become *borderline cases* of function: those where the number of independent variables has been reduced to 0. That's why we said above that the notion of function were an *extension* of that of set.

We have distinguished two moments within the transition from intension to extension: 1) the abstractive moment and 2) the change of identification criterion. In fact, each of those two processes have is own autonomy. This autonomy is a distinctive feature of Weyl's system which is never explicitly explained in *Das Kontinuum*. But Weyl gives a few indications in this direction in some remarks and he confirms this fact in his *Letter to Hölder*. Those two moments are not only distinguishable but they can also be separated in a way.

Explaining this separation, we will be able to lighten the relations between sets and functions in *Das Kontinuum* as for their common nature as well as their distinction.

The common nature of sets and functions

To understand their common nature, we just have to consider individually the second moment of the transition between intension and extension.

Weyl asserts that the identification criterion of *relations*(that is intensions*)* is a purely logical one. It means that the identity between two relations must be decided on solely from their definitions by the way of their logical structure.

At the opposite side, the criterion of identification of sets and functions is irreducible to logical aspects. Weyl asserts that it must be based on an "objective fact". In fact, such an objective fact is given to us by the means of the *intuitive access* we have to basic categories. Thus, this *basic-category intuition* is what is needed to give a *content* to the logical entities. That intuition is the condition to enter in the very domain of mathematics. We can see this is an attitude close to the kantian one.

To summarize, in the mathematical process, takes place the very synthesis of logic and intuition. It permits the constitution of the mathematical universe. This synthesis works in a similar way for sets and functions and explains their common nature.

Processus mathématique

I Intension ⟶ Extension

⇕

Logical Point of view over relations ⟶ (synthesis of logic and intuition) Mathematical Point of view over sets and functions

The origin of the distinction between sets and functions

We can see that the abstraction process can be separated from the change of the identification criterion because the abstraction process step in already at the *intensional level,* that is at the *logical level* of the relations. Indeed, two principles of construction given by Weyl use *explicitly* sets and functions: the *principle of substitution* and the *principle of iteration*. But, in so far as those principles are used to the logical definition of relations, they must be conceived prior to the extensional level. Indeed, the mathematical universe emerges in Weyl's position from a *single* application of the mathematical process.

Thus, how is it possible that sets and functions, which are the extensional entities, step in the use of *intensional* principles?

This problem can be solved easily if we assume the distinction of the abstractive moment and the moment of the identification-criterion change, in the transition from intension to extension. Indeed, Weyl, in a few brief but essential notes, asserts that the sets and functions used for the definitions are not sets and functions in the full sense of those terms[8].

We can see that those particular notions of set and function, which Weyl describes as "purely formal" ones, correspond to *abstract entities* for which the identification criterion is still a logical one. This fact is confirmed by the fact that Weyl forbids the use of the extensional identity for the definition of relations.

[8] [1918], p40 « in a purely formal way » and note n°35. cf also [1919]

This distinction between those two moments (of the transition from intension to extension) shows that the distinction between sets and function in Weyl's system can be found in the only *abstractive* moment before the distinction between intension and extension. Functions are obtained by keeping independent some variables. It amounts to saying that functions are *partly abstracted* entities whereas sets are *totally abstracted* entities.

Let's take the same example as above:

$$R(\underbrace{x_1, \dots, x_k}_{\text{Dependent Variables}} \mid \underbrace{y_1, \dots, y_n}_{\text{Independent Variables}})$$

When we form the function $\check{R}(y_1, \dots, y_n)$, we erase the distinction between all the k-dimensional systems $(x_1, \dots, x_k)$ which are linked to each n-dimensional system $(y_1, \dots, y_n)$. This process by which we erase the distinction between some entities is the essence of the abstraction for Weyl in a quite classic way of thinking. Functions are then partly abstracted in the sense that some *residual variables* are always present after the transition from intension to extension.

The distinction between sets and functions is now clear. We now have to explain why Weyl was compelled to assume, in his mathematical universe, those *partly-abstracted extensional entities* which he calls "functions", that is those kinds of hybrid entities between relations and sets.

Why was Weyl compelled to assume « functions » in his mathematical universe ?

We can found several reasons that have unequal importances.

1) The possibility to apply *partly* the abstractive process supplies a convenient means to "fit" relations into other relations, according to their dimension. In his Letter to Hölder, Weyl gives the example of a 5-dimensional relation $R(u, v \mid x, y, z)$. He shows that the possibility to distinct the independent variables from the dependent ones permits to consider this relation as a "binary relation which depends on the three parameters x, y, z

and is realized between u and v". This relation can then become an object for any relation S(X) whose argument refers to a binary relation. Thus, the possibility to abstract relations is a means for regarding a relation as an object for another relation. And the possibility to apply *partly* this abstractive process permits to make fitting the dimensions of the relations we want to connect.

2) However, there is a deeper reason to the fact that Weyl assumes "functions" in his system. Indeed, it was a necessary condition for Weyl in order to maintain his particular predicativist position, without assuming Russell's RTT.

We have already referred to the fact that Weyl expressed his predicativist position by the way of the *restricted principle*, that is by the prohibition to regard a set as a given totality available for the definition of a new relation. But mathematics, and analysis in particular, can't be developed without the possibility to use, for definition, totalities of ideal objects. This restriction is partly compensated by what I called "arithmetism". Indeed, Weyl's arithmetism permits to regard a countable totality of sets as available for the definition of a new relation, provided that this totality is given by the way of a "finitist" relation.

More precisely, if a totality of sets is given by the means of a *set (of level>1)* then the "restricted principle" forbids us to quantify over the elements of this set. On the opposite, if the totality of sets is given by the way of a relation on which the abstractive process have been partly applied, leaving independent the variable which is used to enumerate the sets, then we can regard this totality of sets as available for definitions, without infringing the restricted principle.

Let's give an example.

We can speak of a real number as a set of rational numbers. If we want to speak about a single real number, then a *set* is enough. For example, $\sqrt{2}$ can be defined as the extension of the property «$x^2<2$ ». On the other hand, we can speak about the series $\sqrt{n}$ only if the argument "n", which is a part of the relation by which the series is defined, have not been abstracted. Thus, we define the relation $R(x|n)$ which means « $x^2<n$ » and we apply the mathematical process, without abstracting the "n". We obtain a function $\check{R}(n)$ which was the series we were looking for. This series is available for the definition of a new relation.

Weyl shows more generally that we have to use this kind of "function" everywhere we have to use a totality of sets for definition. The partial abstraction is then a means to use totalities of sets in a system where the quantification over sets is forbidden.

3) In fact, the situation is more complex because the use of functions and sets, in Weyl's system, becomes fertile only when it is used simultaneously with the *principle of iteration*. Indeed, we could prove the fact below:

> **Sterility of sets and functions without the principle of iteration**
>
> For every basic-objects finitist property P(x), which is defined without the use of the principle of iteration, there is a property P'(x) which is equivalent to P and which is defined without any reference to a set or a function.

(Weyl lets us think that this is a distinctive feature of his system. But he doesn't prove it nor express it clearly. Nevertheless, we can see briefly that it is true on an example.

Let's suppose we have a finitist property P(n) of natural numbers which is defined without the use of the principle of iteration. Let's suppose moreover that the definition of P uses only one function: θ(t). For each t, this function refers to a set of natural numbers.

The only principles that use a function in Weyl's system are the principle of iteration and the principle of substitution. Since we have supposed that the principle of iteration was not used to define P, we must admit that the function θ(t) bas been substituted to a set-variable "X" in the definition of P. But the only primitive relation which uses sets(or functions) in Weyl's system is the membership relation ∈. Therefore, P must be a property logically derived from the two-arguments relation: "(s ∈ θ(t))" without using another function or set (by hypothesis). Now, Weyl's definitionism compels us to admit that the function θ(t) has been obtained by a partial abstractive process over a two-arguments relation of natural numbers: T(u | t) which doesn't use another set or function (by hypothesis). Thus, we know that the relation "(s ∈ θ(t))" is equivalent to T (s , t). Finally, P is equivalent to a property P' where all the occurrences of "(s ∈ θ(t))" have been changed in "T (s , t)". P' is defined without any use of sets or functions. QED The general proof must be just a little more complex)

Owing to this fact, we have to say that the use of the principle of iteration is the last grounds for the assuming of the "functions" in Weyl's system. What we said in 2) (on page 13) is still right. We have to introduce "functions" in Weyl's system in order to use legitimately totality of sets. But this introduction is fertile only because of the principle of iteration. Indeed, without this principle, even the operation of addition between natural numbers is not given in Weyl's system.

Conclusion

We began our research noting a remarkable feature of Weyl's system: sets and functions (the two kinds of extensional entities) co-emerge in the mathematical universe. And they do it in such a way that sets become particular cases of functions.

Why is there such a turning upside out in comparison to Set Theory?

Separating two moments in the transition from intension to extension, we tried to explain the origin of the common nature of sets and functions and the origin of their distinction. It appeared that sets and functions are two kinds of abstract entities whose identification criterion is based on a basic-category intuition. But, unlike sets, functions are *partly abstractive* entities. Why was Weyl compelled to assume those partly abstractive entities?

The predicativist position, when it is expressed only by restricting the universe of the TTR to the level 1, seems not to be able to reconstruct analysis in an appropriate way. Russell was very conscious of it. That's why he adopted his *principle of reducibility*. Weyl judges that it is a mere treachery toward the predicativist position.

Weyl wanted to keep faithful to the predicativist position and chose not to assume other entities than those of level 1. Functions, as partly abstracted entities, became the only way for him to use totalities of sets for definition, without betraying the *restricted principle,* which expressed for Weyl his predicativist position.

But, in fact, this system would have been entirely sterile if this restriction had not been compensated by the position of Weyl I called "arithmetism". The use of "functions" became fruitful because of the *iterative principle* assumed by Weyl. This principle poses a problem because we know, since a work of Kleene[9], that it permits to get over the first level, in the strict sense of the TTR of Russell. Has Weyl unconsciously betrayed the predicativist position too?

I don't know. But at least we have given a reason to understand why Weyl assumed this principle in its particular form. The problem raised by Weyl's principle came from the fact that it permits to assume series of *sets* as complete series. We saw that this is a consequence of what I called Weyl's "arithmetism". The position of Weyl about the natural numbers series is a radical one not only because he assumed that this series is *intuitively* given as a *complete* one (in the particular sense Weyl gives to the word "complete"), but also because he gave to this series a *transcendental* aspect (in the sense I gave above to the word "transcendental"). In other words, Weyl had a very *formal* or *structural* notion of the natural numbers series. And the formal aspect of his notion had great consequences *outside* the natural numbers series. This is confirmed by several passages of Weyl's work. He has probably never used the word "*formal*" for a strategic reason, because he didn't want to create an ambiguity with the *formalist* positions against which he fought.

Therefore, from a purely mathematical and logical point of view, Weyl succeeded in reconstructing mathematical analysis without assuming quantification *over sets*. It is possible because of the use of *functions,* as partly abstracted entities, simultaneously with the particular *arithmetism* he defended. Nevertheless, we have to wonder if this radical position about the natural numbers series is more legitimated than the principle of reducibility itself.

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[9] cf [F1988]